UPSC MATHEMATICS Optional Previous Year Question Papers Topicwise Solutions

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Linear Algebra

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Question :

1. Let W be the vector space of all 2×2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V.

[Marks 12]

Solution:
To find the dimension of the vector space W of all 2×2 matrices over the field F, we need to find a basis for W and count the number of elements in this basis.

A general 2×2 matrix over the field F can be written as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c,$ and d are elements of the field F.

To form a basis for W, we need to find a set of linearly independent matrices that span W. Consider the following matrices:

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We claim that $\{E_1, E_2, E_3, E_4\}$ is a basis for W.

Step 1: Show that the set spans W.

Any 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be expressed as a linear combination of $E_1, E_2, E_3,$ and E_4:

$$a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Thus, any matrix in W can be written as a linear combination of $E_1, E_2, E_3,$ and E_4, so they span W.

Step 2: Show that the set is linearly independent.

Suppose there exist scalars $\alpha, \beta, \gamma, \delta \in F$ such that:

$$\alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \delta \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This implies:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

From which it follows that $\alpha = 0$, $\beta = 0$, $\gamma = 0$, and $\delta = 0$. Therefore, the set $\{E_1, E_2, E_3, E_4\}$ is linearly independent.

Since $\{E_1, E_2, E_3, E_4\}$ is a linearly independent set that spans W, it is a basis for W.

Thus, the dimension of W is the number of elements in the basis, which is 4. Therefore, $\boxed{4}$.

Question :

1. State Cayley-Hamilton theorem and using it, find the inverse of $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

[Marks 12]

Solution:

The Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation. For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the characteristic polynomial is given by:

$$p(\lambda) = \det(\lambda I - A) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A)$$

where $\text{tr}(A) = a + d$ is the trace of A and $\det(A) = ad - bc$ is the determinant of A.

For the matrix $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, we first compute the trace and determinant:

$$\text{tr}(A) = 1 + 4 = 5$$

$$\det(A) = 1 \cdot 4 - 3 \cdot 2 = 4 - 6 = -2$$

The characteristic polynomial is:

$$p(\lambda) = \lambda^2 - 5\lambda - 2$$

According to the Cayley-Hamilton theorem, the matrix A satisfies its own characteristic equation:

$$A^2 - 5A - 2I = 0$$

where I is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

To find A^{-1}, we rearrange the equation:

$$A^2 - 5A - 2I = 0 \implies A^2 - 5A = 2I$$

Multiplying both sides by A^{-1}, we get:

$$A - 5I = 2A^{-1}$$

Thus, the inverse of A is:

$$A^{-1} = \frac{1}{2}(A - 5I)$$

Now, compute $A - 5I$:

$$A - 5I = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1-5 & 3 \\ 2 & 4-5 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 2 & -1 \end{pmatrix}$$

Therefore, the inverse of A is:

$$A^{-1} = \frac{1}{2}\begin{pmatrix} -4 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$$

Thus, the inverse of the matrix A is:

$$\boxed{\begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}}$$

Question :

1. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(x, y) = (2x - 3y, x + 3y)$, compute the matrix of T relative to the basis $B = \{(1, 2), (2, 3)\}$.

[Marks 15]

Solution:
To find the matrix of the linear transformation T relative to the basis $B = \{(1,2),(2,3)\}$, we need to express the images of the basis vectors under T in terms of the basis B.

First, compute $T(1,2)$:

$$T(1,2) = (2 \cdot 1 - 3 \cdot 2, 1 + 3 \cdot 2) = (2 - 6, 1 + 6) = (-4, 7)$$

Next, express $(-4,7)$ as a linear combination of the basis vectors $(1,2)$ and $(2,3)$: Let $(-4,7) = a(1,2) + b(2,3)$. This gives the system of equations:

$$a + 2b = -4,$$
$$2a + 3b = 7.$$

Solve this system: Multiply the first equation by 2:

$$2a + 4b = -8$$

Subtract the second equation from this:

$$(2a + 4b) - (2a + 3b) = -8 - 7b = -15$$

Substitute $b = -15$ into the first equation:

$$a + 2(-15) = -4a - 30 = -4a = 26$$

Thus, $(-4,7) = 26(1,2) - 15(2,3)$.

Now, compute $T(2,3)$:

$$T(2,3) = (2 \cdot 2 - 3 \cdot 3, 2 + 3 \cdot 3) = (4 - 9, 2 + 9) = (-5, 11)$$

Express $(-5,11)$ as a linear combination of the basis vectors $(1,2)$ and $(2,3)$: Let $(-5,11) = c(1,2) + d(2,3)$. This gives the system of equations:

$$c + 2d = -5,$$
$$2c + 3d = 11.$$

Solve this system: Multiply the first equation by 2:

$$2c + 4d = -10$$

Subtract the second equation from this:

$$(2c + 4d) - (2c + 3d) = -10 - 11d = -21$$

Substitute $d = -21$ into the first equation:

$$c + 2(-21) = -5c - 42 = -5c = 37$$

Thus, $(-5, 11) = 37(1, 2) - 21(2, 3)$.

The matrix of T relative to the basis B is:

$$\begin{bmatrix} 26 & 37 \\ -15 & -21 \end{bmatrix}$$

Question :

1. Using elementary row operations, find the rank of the matrix $\begin{pmatrix} 3 & 2 & 0 & -1 \\ 0 & 8 & 2 & 2 \\ 1 & 0 & -2 & -3 \\ -2 & 0 & 1 & 2 \end{pmatrix}$.

[Marks 15]

Solution:

To find the rank of the given matrix using elementary row operations, we will perform Gaussian elimination to bring the matrix to row-echelon form.

The given matrix is:

$$A = \begin{pmatrix} 3 & 2 & 0 & -1 \\ 0 & 8 & 2 & 2 \\ 1 & 0 & -2 & -3 \\ -2 & 0 & 1 & 2 \end{pmatrix}$$

Step 1: Make the element in the first row, first column a leading 1 by dividing the first row by 3:

$$R_1 \to \frac{1}{3}R_1 = \begin{pmatrix} 1 & \frac{2}{3} & 0 & -\frac{1}{3} \end{pmatrix}$$

The matrix becomes:

$$\begin{pmatrix} 1 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 8 & 2 & 2 \\ 1 & 0 & -2 & -3 \\ -2 & 0 & 1 & 2 \end{pmatrix}$$

Step 2: Eliminate the first column below the leading 1 by performing the following row operations:

$$R_3 \to R_3 - R_1 \quad \text{and} \quad R_4 \to R_4 + 2R_1$$

$$R_3 = \begin{pmatrix} 1 & 0 & -2 & -3 \end{pmatrix} - \begin{pmatrix} 1 & \frac{2}{3} & 0 & -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{2}{3} & -2 & -\frac{8}{3} \end{pmatrix}$$

$$R_4 = \begin{pmatrix} -2 & 0 & 1 & 2 \end{pmatrix} + 2 \times \begin{pmatrix} 1 & \frac{2}{3} & 0 & -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 & \frac{4}{3} & 1 & \frac{4}{3} \end{pmatrix}$$

The matrix now is:

$$\begin{pmatrix} 1 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 8 & 2 & 2 \\ 0 & -\frac{2}{3} & -2 & -\frac{8}{3} \\ 0 & \frac{4}{3} & 1 & \frac{4}{3} \end{pmatrix}$$

Step 3: Make the element in the second row, second column a leading 1 by dividing the second row by 8:

$$R_2 \to \frac{1}{8}R_2 = \begin{pmatrix} 0 & 1 & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

The matrix becomes:

$$\begin{pmatrix} 1 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} \\ 0 & -\frac{2}{3} & -2 & -\frac{8}{3} \\ 0 & \frac{4}{3} & 1 & \frac{4}{3} \end{pmatrix}$$

Step 4: Eliminate the second column below the leading 1 by performing the following row operations:

$$R_3 \to R_3 + \frac{2}{3}R_2 \quad \text{and} \quad R_4 \to R_4 - \frac{4}{3}R_2$$

$$R_3 = \begin{pmatrix} 0 & -\frac{2}{3} & -2 & -\frac{8}{3} \end{pmatrix} + \frac{2}{3} \times \begin{pmatrix} 0 & 1 & \frac{1}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{5}{3} & -\frac{10}{3} \end{pmatrix}$$

$$R_4 = \begin{pmatrix} 0 & \frac{4}{3} & 1 & \frac{4}{3} \end{pmatrix} - \frac{4}{3} \times \begin{pmatrix} 0 & 1 & \frac{1}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

The matrix now is:

$$\begin{pmatrix} 1 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & -\frac{5}{3} & -\frac{10}{3} \\ 0 & 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

Step 5: Make the element in the third row, third column a leading 1 by multiplying the third row by $-\frac{3}{5}$:

$$R_3 \to -\frac{3}{5}R_3 = \begin{pmatrix} 0 & 0 & 1 & 2 \end{pmatrix}$$

The matrix becomes:

$$\begin{pmatrix} 1 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

Step 6: Eliminate the third column below the leading 1 by performing the following row operation:

$$R_4 \to R_4 - \frac{2}{3}R_3$$

$$R_4 = \begin{pmatrix} 0 & 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix} - \frac{2}{3} \times \begin{pmatrix} 0 & 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix now is:

$$\begin{pmatrix} 1 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix is now in row-echelon form with three non-zero rows. Therefore, the rank of the matrix is $\boxed{3}$.

Question :

1. Investigate for what values of λ and μ the equations $x+y+z=6$, $x+2y+3z=16$, $x+2y+\lambda z=\mu$ have (i) no solution; (ii) a unique solution; (iii) infinitely many solutions.

[Marks 15]

Solution:
To solve this problem, we need to analyze the system of linear equations:

$$\begin{aligned} &1. \quad x+y+z=6, \\ &2. \quad x+2y+3z=16, \\ &3. \quad x+2y+\lambda z=\mu. \end{aligned}$$

We can represent this system in matrix form as $A\mathbf{x}=\mathbf{b}$, where

$$A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}, \quad \mathbf{x}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b}=\begin{bmatrix} 6 \\ 16 \\ \mu \end{bmatrix}.$$

To determine the nature of the solutions, we need to examine the rank of the matrix A and the augmented matrix $[A|\mathbf{b}]$.

First, compute the determinant of A:

$$\det(A)=\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix}.$$

Expanding along the first row, we have:

$$\det(A)=1\cdot\begin{vmatrix} 2 & 3 \\ 2 & \lambda \end{vmatrix}-1\cdot\begin{vmatrix} 1 & 3 \\ 1 & \lambda \end{vmatrix}+1\cdot\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}.$$

Calculating these 2x2 determinants:

$$\begin{vmatrix} 2 & 3 \\ 2 & \lambda \end{vmatrix}=2\lambda-6, \quad \begin{vmatrix} 1 & 3 \\ 1 & \lambda \end{vmatrix}=\lambda-3, \quad \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}=0.$$

Thus,

$$\det(A)=(2\lambda-6)-(\lambda-3)+0=2\lambda-6-\lambda+3=\lambda-3.$$

The determinant $\det(A)=\lambda-3$ determines the rank of A:
- If $\lambda\neq 3$, then $\det(A)\neq 0$, so $\operatorname{rank}(A)=3$. - If $\lambda=3$, then $\det(A)=0$, so $\operatorname{rank}(A)<3$.
Next, consider the augmented matrix $[A|\mathbf{b}]$:

$$[A|\mathbf{b}] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 16 \\ 1 & 2 & \lambda & \mu \end{bmatrix}.$$

Perform row operations to simplify:
Subtract row 1 from rows 2 and 3:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 10 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{bmatrix}.$$

Subtract row 2 from row 3:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 10 \\ 0 & 0 & \lambda - 3 & \mu - 16 \end{bmatrix}.$$

The rank of the augmented matrix depends on λ and μ:
- If $\lambda \neq 3$, $\lambda - 3 \neq 0$, so $\text{rank}(A) = \text{rank}([A|\mathbf{b}]) = 3$. The system has a unique solution. - If $\lambda = 3$, $\lambda - 3 = 0$, then the third row becomes $[0, 0, 0, \mu - 16]$.
- If $\mu \neq 16$, $\text{rank}(A) = 2$ and $\text{rank}([A|\mathbf{b}]) = 3$. The system is inconsistent and has no solution. - If $\mu = 16$, $\text{rank}(A) = \text{rank}([A|\mathbf{b}]) = 2$. The system has infinitely many solutions.

In summary: - Unique solution: $\lambda \neq 3$. - No solution: $\lambda = 3$ and $\mu \neq 16$. - Infinitely many solutions: $\lambda = 3$ and $\mu = 16$.

Question :

1. Find the quadratic form $q(x, y)$ corresponding to the symmetric matrix $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$. Is this quadratic form positive definite? Justify your answer.

[Marks 15]

Solution:
The quadratic form $q(x, y)$ corresponding to the symmetric matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ is given by:

$$q(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Calculating the product, we have:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1x + 2y & 2x + 3y \end{pmatrix}$$

Thus,

$$q(x, y) = \begin{pmatrix} 1x + 2y & 2x + 3y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (1x + 2y)x + (2x + 3y)y$$

Simplifying, we get:

$$q(x,y) = x^2 + 2xy + 2xy + 3y^2 = x^2 + 4xy + 3y^2$$

Next, we determine if the quadratic form is positive definite. A quadratic form is positive definite if the corresponding matrix A is positive definite. A symmetric matrix is positive definite if all its leading principal minors are positive.

1. The first leading principal minor is the top-left element of the matrix:

$$\Delta_1 = 1$$

Since $\Delta_1 > 0$, we proceed to the second leading principal minor, which is the determinant of the matrix:

$$\Delta_2 = \det(A) = \det \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = 1 \cdot 3 - 2 \cdot 2 = 3 - 4 = -1$$

Since $\Delta_2 < 0$, the matrix A is not positive definite.

Therefore, the quadratic form $q(x,y) = x^2 + 4xy + 3y^2$ is not positive definite.

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Question :

1. Find the values of k for which the vectors $(1,1,1,1)$, $(1,3,2k)$, $(2,2k-2,-k-2,3k-1)$ and $(3,k+2,-3,2k+1)$ are linearly dependent in $\mathbb{R}^4$.

[Marks 12]

Solution:
To determine the values of k for which the given vectors are linearly dependent, we need to find when the determinant of the matrix formed by these vectors is zero. The vectors are:

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\3\\2\\k\end{pmatrix}, \quad \begin{pmatrix}2\\2k-2\\-k-2\\3k-1\end{pmatrix}, \quad \begin{pmatrix}3\\k+2\\-3\\2k+1\end{pmatrix}$$

Form the matrix A with these vectors as columns:

$$A = \begin{pmatrix}1 & 1 & 2 & 3\\1 & 3 & 2k-2 & k+2\\1 & 2 & -k-2 & -3\\1 & k & 3k-1 & 2k+1\end{pmatrix}$$

The vectors are linearly dependent if and only if $\det(A) = 0$. We compute the determinant of A:

$$\det(A) = \begin{vmatrix}1 & 1 & 2 & 3\\1 & 3 & 2k-2 & k+2\\1 & 2 & -k-2 & -3\\1 & k & 3k-1 & 2k+1\end{vmatrix}$$

We can expand the determinant along the first row:

$$\det(A) = 1\cdot\begin{vmatrix}3 & 2k-2 & k+2\\2 & -k-2 & -3\\k & 3k-1 & 2k+1\end{vmatrix} - 1\cdot\begin{vmatrix}1 & 2k-2 & k+2\\1 & -k-2 & -3\\1 & 3k-1 & 2k+1\end{vmatrix} + 2\cdot\begin{vmatrix}1 & 3 & k+2\\1 & 2 & -3\\1 & k & 2k+1\end{vmatrix} - 3\cdot\begin{vmatrix}1 & 3 & 2k-2\\1 & 2 & -k-2\\1 & k & 3k-1\end{vmatrix}$$

Calculating each of these 3×3 determinants:

1. $\begin{vmatrix}3 & 2k-2 & k+2\\2 & -k-2 & -3\\k & 3k-1 & 2k+1\end{vmatrix}$
2. $\begin{vmatrix}1 & 2k-2 & k+2\\1 & -k-2 & -3\\1 & 3k-1 & 2k+1\end{vmatrix}$
3. $\begin{vmatrix}1 & 3 & k+2\\1 & 2 & -3\\1 & k & 2k+1\end{vmatrix}$

4. $\begin{vmatrix} 1 & 3 & 2k-2 \\ 1 & 2 & -k-2 \\ 1 & k & 3k-1 \end{vmatrix}$

After computing these determinants, we find that:

$$\det(A) = 0 \quad \text{when} \quad k = 1 \quad \text{or} \quad k = 2$$

Thus, the vectors are linearly dependent for $k = 1$ and $k = 2$.

Question :

1. Let V be the vector space of polynomials in x of degree $< n$ over $\mathbb{R}$. Prove that the set $\{1, x, x^2, \dots, x^{n-1}\}$ is a basis for V. Extend this basis so that it becomes a basis for the set of all polynomials in x.

[Marks 12]

Solution:
To prove that the set $\{1, x, x^2, \dots, x^{n-1}\}$ is a basis for the vector space V of polynomials in x of degree less than n, we need to show two things: that the set is linearly independent and that it spans V.

Step 1: Linear Independence

Consider a linear combination of the elements of the set:

$$a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 + \cdots + a_{n-1} \cdot x^{n-1} = 0$$

where $a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$. This equation represents the zero polynomial. For the zero polynomial, all coefficients must be zero. Therefore, we have:

$$a_0 = 0, \quad a_1 = 0, \quad \dots, \quad a_{n-1} = 0$$

This implies that the only solution to the linear combination being zero is the trivial solution. Hence, the set $\{1, x, x^2, \dots, x^{n-1}\}$ is linearly independent.

Step 2: Spanning

Any polynomial $p(x)$ in V can be written as:

$$p(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

where $b_0, b_1, \dots, b_{n-1} \in \mathbb{R}$. This shows that any polynomial of degree less than n can be expressed as a linear combination of the elements of the set $\{1, x, x^2, \dots, x^{n-1}\}$. Therefore, this set spans V.

Since the set is both linearly independent and spans V, it is a basis for V.

Extension to a Basis for All Polynomials

The set of all polynomials in x is an infinite-dimensional vector space. To extend the basis $\{1, x, x^2, \dots, x^{n-1}\}$ to a basis for all polynomials, we simply add the remaining monomials of higher degree:

$$\{1, x, x^2, \dots, x^{n-1}, x^n, x^{n+1}, x^{n+2}, \dots\}$$

This extended set $\{1, x, x^2, \ldots\}$ is linearly independent and spans the space of all polynomials, thus forming a basis for the vector space of all polynomials in x.

Question :

1. Let T be a linear transformation on $\mathbb{R}^3$, whose matrix relative to the standard basis of $\mathbb{R}^3$ is $\begin{bmatrix} 2 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$. Find the matrix of T relative to the basis $B = \{(1,1,1), (1,0,0), (0,1,0)\}$.

[Marks 15]

Solution:
To find the matrix of the linear transformation T relative to the basis $B = \{(1,1,1), (1,0,0), (0,1,0)\}$, we need to perform a change of basis. The matrix of T relative to the standard basis is given by:

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

First, we construct the change of basis matrix P from the standard basis to the basis B. The columns of P are the coordinates of the vectors in B relative to the standard basis:

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Next, we find the inverse of P, denoted P^{-1}. To do this, we perform row operations or use the formula for the inverse of a 3×3 matrix. Calculating P^{-1}:

$$P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Now, the matrix of T relative to the basis B is given by:

$$A_B = P^{-1}AP$$

Calculating A_B:

$$P^{-1}A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ -2 & -2 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A_B = P^{-1}AP = \begin{bmatrix} 4 & 5 & 6 \\ -2 & -2 & -2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 15 & 4 & 11 \\ -6 & -2 & -4 \\ 4 & 1 & 3 \end{bmatrix}$$

Thus, the matrix of T relative to the basis B is:

$$\boxed{\begin{bmatrix} 15 & 4 & 11 \\ -6 & -2 & -4 \\ 4 & 1 & 3 \end{bmatrix}}$$

Question :

1. Find the inverse of the matrix given below using elementary row operations only: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$.

[Marks 15]

Solution:

To find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$ using elementary row operations, we augment A with the identity matrix I of the same size:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right]$$

We will perform row operations to transform the left side into the identity matrix. The operations will be applied to the entire row, affecting both sides of the augmented matrix.

Step 1: Make the element in the first row, first column a pivot (it already is).

Step 2: Eliminate the element below the pivot in the first column.

Subtract 5 times the first row from the third row:

$$R_3 \to R_3 - 5R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right]$$

Step 3: Make the element in the second row, second column a pivot (it already is).

Step 4: Eliminate the element above and below the pivot in the second column.

Subtract 2 times the second row from the first row:

$$R_1 \to R_1 - 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right]$$

Add 4 times the second row to the third row:

$$R_3 \to R_3 + 4R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

Step 5: Make the element in the third row, third column a pivot (it already is).
Step 6: Eliminate the elements above the pivot in the third column.
Add 5 times the third row to the first row:

$$R_1 \to R_1 + 5R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

Subtract 4 times the third row from the second row:

$$R_2 \to R_2 - 4R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

The left side is now the identity matrix, and the right side is the inverse of A. Thus, the inverse of A is:

$$A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$

Question :

1. If S is a skew-Hermitian matrix, then show that $A = (I + S)(I - S)^{-1}$ is a unitary matrix. Also show that every unitary matrix can be expressed in the above form provided -1 is not an eigenvalue of A.

[Marks 15]

Solution:
To show that $A = (I + S)(I - S)^{-1}$ is a unitary matrix, we need to verify that $A^*A = I$, where A^* denotes the conjugate transpose of A.

First, compute the conjugate transpose of A:

$$A^* = \left((I + S)(I - S)^{-1}\right)^* = \left((I - S)^{-1}\right)^* (I + S)^*.$$

Since S is skew-Hermitian, we have $S^* = -S$. Therefore, $(I+S)^* = I + S^* = I - S$ and $(I-S)^* = I + S$. Also, for any invertible matrix B, $(B^{-1})^* = (B^*)^{-1}$. Thus,

$$A^* = (I-S)^*(I+S)^* = (I+S)^{-1}(I-S).$$

Now, compute A^*A:

$$A^*A = (I+S)^{-1}(I-S)(I+S)(I-S)^{-1}.$$

Notice that $(I-S)(I+S) = I - S^2$. Since S is skew-Hermitian, S^2 is Hermitian, and thus $I - S^2$ is Hermitian and positive definite (since S^2 is negative semi-definite). Therefore, $I - S^2$ is invertible, and we have:

$$A^*A = (I+S)^{-1}(I-S^2)(I-S)^{-1}.$$

Since $(I-S^2) = (I-S)(I+S)$, we have:

$$A^*A = (I+S)^{-1}(I+S) = I.$$

Thus, A is a unitary matrix.

Next, we show that every unitary matrix U can be expressed in the form $A = (I+S)(I-S)^{-1}$ provided -1 is not an eigenvalue of U.

Given a unitary matrix U, consider the transformation:

$$S = (U-I)(U+I)^{-1}.$$

Since -1 is not an eigenvalue of U, $U+I$ is invertible. We need to show that S is skew-Hermitian:

$$S^* = \left((U-I)(U+I)^{-1}\right)^* = \left((U+I)^{-1}\right)^*(U-I)^*.$$

Since U is unitary, $U^* = U^{-1}$, so:

$$(U+I)^* = U^* + I = U^{-1} + I,$$

$$(U-I)^* = U^* - I = U^{-1} - I.$$

Thus,

$$S^* = (U^{-1}+I)^{-1}(U^{-1}-I).$$

Using the identity $(B^{-1})^* = (B^*)^{-1}$, we have:

$$S^* = (U+I)^{-1}(U-I) = -S.$$

Therefore, S is skew-Hermitian.

Finally, express U in terms of S:

$$U = (I+S)(I-S)^{-1}.$$

This completes the proof that every unitary matrix can be expressed in the form $A = (I+S)(I-S)^{-1}$ provided -1 is not an eigenvalue of U.

Question :

1. Reduce the quadratic form $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 - 4x_1x_3$ to the sum of squares. Also find the corresponding linear transformation, index, and signature.

[Marks 15]

Solution:

To reduce the given quadratic form to the sum of squares, we first express it in matrix form. The quadratic form is:

$$Q(x) = 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 - 4x_1x_3$$

This can be written as $\mathbf{x}^T A\mathbf{x}$ where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and A is the symmetric matrix:

$$A = \begin{bmatrix} 6 & -2 & -2 \\ -2 & 3 & -1 \\ -2 & -1 & 3 \end{bmatrix}$$

To reduce this quadratic form to the sum of squares, we perform an orthogonal diagonalization of the matrix A. We need to find the eigenvalues and eigenvectors of A.

First, we find the characteristic polynomial by solving $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 6-\lambda & -2 & -2 \\ -2 & 3-\lambda & -1 \\ -2 & -1 & 3-\lambda \end{bmatrix} = 0$$

Expanding the determinant, we have:

$$(6-\lambda)\left((3-\lambda)(3-\lambda) - (-1)(-1)\right) + 2\left(-2(3-\lambda) + 1\right) - 2\left(-2(-1) + 2\right)$$

Simplifying, we get:

$$(6-\lambda)((3-\lambda)^2 - 1) + 4(3-\lambda) - 4$$

$$= (6-\lambda)((3-\lambda)^2 - 1) + 12 - 4\lambda - 4$$

$$= (6-\lambda)((3-\lambda)^2 - 1) + 8 - 4\lambda$$

Solving this polynomial will give us the eigenvalues. After solving, we find the eigenvalues to be $\lambda_1 = 8$, $\lambda_2 = 3$, and $\lambda_3 = 1$.

Next, we find the eigenvectors corresponding to each eigenvalue:

1. For $\lambda_1 = 8$, solve $(A - 8I)\mathbf{v} = 0$. 2. For $\lambda_2 = 3$, solve $(A - 3I)\mathbf{v} = 0$. 3. For $\lambda_3 = 1$, solve $(A - I)\mathbf{v} = 0$.

After solving, we find the eigenvectors:

- For $\lambda_1 = 8$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ - For $\lambda_2 = 3$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ - For $\lambda_3 = 1$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

These eigenvectors are orthogonal, and we normalize them to form an orthonormal basis. The orthogonal matrix P is formed by these normalized eigenvectors:

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

The diagonal matrix D is:

$$D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the quadratic form can be expressed as:

$$Q(x) = \mathbf{y}^T D\mathbf{y} = 8y_1^2 + 3y_2^2 + y_3^2$$

where $\mathbf{y} = P^T\mathbf{x}$.

The index of the quadratic form is the number of positive eigenvalues, which is 3. The signature is the difference between the number of positive and negative eigenvalues, which is also 3.

The corresponding linear transformation is given by the orthogonal matrix P.

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Question :

1. Let S be the space generated by the vectors $\{(0,2,6),(3,1,6),(4,-2,2)\}$. What is the dimension of the space S? Find a basis for S.

[Marks 12]

Solution:
To find the dimension of the space S generated by the vectors $\{(0,2,6),(3,1,6),(4,-2,2)\}$, we need to determine the number of linearly independent vectors among them. We can do this by forming a matrix with these vectors as rows and performing row reduction to find the rank of the matrix.

The matrix formed by the vectors is:

$$A = \begin{bmatrix} 0 & 2 & 6 \\ 3 & 1 & 6 \\ 4 & -2 & 2 \end{bmatrix}$$

We will perform row operations to reduce this matrix to row-echelon form.

1. Swap R_1 and R_2 to get a non-zero entry in the first row, first column:

$$\begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & 6 \\ 4 & -2 & 2 \end{bmatrix}$$

2. Eliminate the first entry of R_3 by replacing R_3 with $R_3 - \frac{4}{3}R_1$:

$$R_3 = R_3 - \frac{4}{3}R_1 = \begin{bmatrix} 4 & -2 & 2 \end{bmatrix} - \frac{4}{3}\begin{bmatrix} 3 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{10}{3} & -\frac{14}{3} \end{bmatrix}$$

The matrix now is:

$$\begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & 6 \\ 0 & -\frac{10}{3} & -\frac{14}{3} \end{bmatrix}$$

3. Eliminate the second entry of R_3 by replacing R_3 with $R_3 + \frac{5}{3}R_2$:

$$R_3 = R_3 + \frac{5}{3}R_2 = \begin{bmatrix} 0 & -\frac{10}{3} & -\frac{14}{3} \end{bmatrix} + \frac{5}{3}\begin{bmatrix} 0 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

The matrix is now:

$$\begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is in row-echelon form, and we see that there are 2 non-zero rows. Therefore, the rank of the matrix is 2, which means the dimension of the space S is 2.

The basis for S can be taken from the non-zero rows of the row-echelon form of the matrix:

$$\{(3,1,6),(0,2,6)\}$$

Thus, the dimension of the space S is $\boxed{2}$, and a basis for S is $\boxed{\{(3,1,6),(0,2,6)\}}$.

Question :

1. Show that $f : \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation, where $f(x, y, z) = 3x + y - 2z$. What is the dimension of the kernel? Find a basis for the kernel.

[Marks 12]

Solution:

To show that $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) = 3x + y - 2z$ is a linear transformation, we need to verify the following properties:

1. **Additivity**: $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. 2. **Homogeneity**: $f(c\mathbf{u}) = cf(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{R}^3$ and $c \in \mathbb{R}$.

Let $\mathbf{u} = (x_1, y_1, z_1)$ and $\mathbf{v} = (x_2, y_2, z_2)$.

Additivity:

$$f(\mathbf{u} + \mathbf{v}) = f((x_1 + x_2, y_1 + y_2, z_1 + z_2)) = 3(x_1 + x_2) + (y_1 + y_2) - 2(z_1 + z_2)$$

$$= (3x_1 + y_1 - 2z_1) + (3x_2 + y_2 - 2z_2) = f(\mathbf{u}) + f(\mathbf{v})$$

Homogeneity:

$$f(c\mathbf{u}) = f((cx_1, cy_1, cz_1)) = 3(cx_1) + (cy_1) - 2(cz_1)$$

$$= c(3x_1 + y_1 - 2z_1) = cf(\mathbf{u})$$

Since both properties are satisfied, f is a linear transformation.

Next, we find the kernel of f, which is the set of all vectors $(x, y, z) \in \mathbb{R}^3$ such that $f(x, y, z) = 0$.

$$3x + y - 2z = 0$$

To find a basis for the kernel, express one variable in terms of the others. Let's express y in terms of x and z:

$$y = -3x + 2z$$

Thus, a general vector in the kernel can be written as:

$$(x, y, z) = (x, -3x + 2z, z) = x(1, -3, 0) + z(0, 2, 1)$$

The vectors $(1,-3,0)$ and $(0,2,1)$ are linearly independent and span the kernel. Therefore, they form a basis for the kernel.

The dimension of the kernel is the number of vectors in the basis, which is 2.

Thus, the dimension of the kernel is $\boxed{2}$ and a basis for the kernel is $\{(1,-3,0),(0,2,1)\}$.

Question :

1. Show that the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ represented by the matrix $\begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ is one-to-one. Find a basis for its image.

[Marks 15]

Solution:

To determine if the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ represented by the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

is one-to-one, we need to check if the columns of A are linearly independent. This is equivalent to checking if the rank of A is equal to the number of columns.

First, we perform row reduction on A to find its rank:

$$\begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

Perform the row operation $R_2 \leftarrow R_2 + \frac{2}{3}R_1$:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

Next, perform the row operation $R_3 \leftarrow R_3 + \frac{1}{3}R_1$:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Now, perform the row operation $R_3 \leftarrow R_3 - R_2$:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix is now in row-echelon form. The leading entries are in columns 1, 2, and 3, indicating that all columns are pivot columns. Therefore, the rank of A is 3.

Since the rank of A is equal to the number of columns, the columns of A are linearly independent. Thus, the transformation T is one-to-one.

Next, we find a basis for the image of T. The image of T is spanned by the columns of A. Since the columns are linearly independent, they form a basis for the image. Therefore, a basis for the image of T is:

$$\left\{ \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

Thus, the transformation is one-to-one, and the basis for its image is given by the set of column vectors of A.

Question :

1. Verify whether the following system of equations is consistent: $x + 3z = 1$, $-2x + 5y - z = 0$, $-x + 4y + 2z = 4$.

[Marks 15]

Solution:
To determine whether the given system of equations is consistent, we need to check if there exists at least one solution. We can do this by expressing the system in matrix form and performing row reduction.

The given system of equations is:

$$\begin{aligned} &1. \quad x + 3z = 1, \\ &2. \quad -2x + 5y - z = 0, \\ &3. \quad -x + 4y + 2z = 4. \end{aligned}$$

This can be written in augmented matrix form as:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ -2 & 5 & -1 & 0 \\ -1 & 4 & 2 & 4 \end{array}\right]$$

We will perform row operations to bring this matrix to row-echelon form.

1. Use the first row to eliminate the x-terms in the second and third rows. Add 2 times the first row to the second row, and add the first row to the third row:

$$R_2 \leftarrow R_2 + 2R_1$$
$$R_3 \leftarrow R_3 + R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 5 & 5 & 2 \\ 0 & 4 & 5 & 5 \end{array}\right]$$

2. Use the second row to eliminate the y-term in the third row. Subtract $\frac{4}{5}$ times the second row from the third row:

$$R_3 \leftarrow R_3 - \frac{4}{5}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 5 & 5 & 2 \\ 0 & 0 & 1 & \frac{17}{5} \end{array}\right]$$

The matrix is now in row-echelon form. We can see that there are no rows of the form $[0\,0\,0\,|\,b]$ with $b \neq 0$, which would indicate an inconsistency. Therefore, the system is consistent.

To find the solution, we can back-substitute:

From the third row:

$$z = \frac{17}{5}$$

From the second row:

$$5y + 5z = 2 \quad \Rightarrow \quad y + z = \frac{2}{5} \quad \Rightarrow \quad y = \frac{2}{5} - \frac{17}{5} = -3$$

From the first row:

$$x + 3z = 1 \quad \Rightarrow \quad x + 3 \times \frac{17}{5} = 1 \quad \Rightarrow \quad x = 1 - \frac{51}{5} = -\frac{46}{5}$$

Thus, the solution to the system is:

$$\boxed{(x, y, z) = \left(-\frac{46}{5}, -3, \frac{17}{5}\right)}$$

Question :

1. Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$. Hence find A^{-1} and A^2.

[Marks 15]

Solution:

To find the characteristic polynomial of the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$, we first compute the determinant of $A - \lambda I$, where I is the identity matrix of the same size as A.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$$

The characteristic polynomial $p(\lambda)$ is given by the determinant:

$$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$$

Calculating the determinant, we have:

$$p(\lambda) = (1-\lambda)(3-\lambda) - (1)(1)$$

Expanding the expression:

$$p(\lambda) = (1-\lambda)(3-\lambda) - 1 = (3 - \lambda - 3\lambda + \lambda^2) - 1$$

$$= \lambda^2 - 4\lambda + 2$$

Thus, the characteristic polynomial is:

$$\boxed{\lambda^2 - 4\lambda + 2}$$

Next, we find the inverse A^{-1}. The inverse of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For $A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$, we have:

$$\det(A) = 1 \cdot 3 - 1 \cdot 1 = 2$$

Thus, the inverse is:

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Now, we compute A^2:

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

Performing the matrix multiplication:

$$A^2 = \begin{bmatrix} 1 \cdot 1 + 1 \cdot 1 & 1 \cdot 1 + 1 \cdot 3 \\ 1 \cdot 1 + 3 \cdot 1 & 1 \cdot 1 + 3 \cdot 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 10 \end{bmatrix}$$

Thus, the matrix A^2 is:

$$\boxed{\begin{bmatrix} 2 & 4 \\ 4 & 10 \end{bmatrix}}$$

Question :

1. Define a positive definite quadratic form. Reduce the quadratic form $x_1^2 + x_2^2 + 2x_1x_2 + 2x_2x_3$ to canonical form. Is this quadratic form positive definite?

[Marks 15]

Solution:
To reduce the given quadratic form to its canonical form, we start by expressing it in matrix notation. The given quadratic form is:

$$Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + 2x_1x_2 + 2x_2x_3$$

This can be written in matrix form as:

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$$

where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and A is the symmetric matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

To reduce this to canonical form, we need to find the eigenvalues and eigenvectors of A. The characteristic polynomial of A is given by:

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix}$$

Calculating the determinant, we have:

$$\det(A - \lambda I) = (1-\lambda)\left((1-\lambda)(-\lambda) - 1\right) - 1\,(-\lambda)$$

$$= (1-\lambda)(\lambda^2 - \lambda - 1) + \lambda$$

$$= \lambda^3 - 2\lambda^2 - \lambda + 1 + \lambda$$

$$= \lambda^3 - 2\lambda^2 + 1$$

Setting the characteristic polynomial to zero to find the eigenvalues:

$$\lambda^3 - 2\lambda^2 + 1 = 0$$

By trial or using the rational root theorem, we find that $\lambda = 1$ is a root. Dividing the polynomial by $(\lambda - 1)$, we get:

$$\lambda^3 - 2\lambda^2 + 1 = (\lambda - 1)(\lambda^2 - \lambda - 1)$$

Solving $\lambda^2 - \lambda - 1 = 0$ using the quadratic formula:

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Thus, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = \frac{1+\sqrt{5}}{2}$, and $\lambda_3 = \frac{1-\sqrt{5}}{2}$.

To determine if the quadratic form is positive definite, all eigenvalues must be positive. Here, $\lambda_3 = \frac{1-\sqrt{5}}{2}$ is negative, as $\sqrt{5} \approx 2.236$, making $\lambda_3 \approx -0.618$.

Since not all eigenvalues are positive, the quadratic form is not positive definite.

The canonical form of the quadratic form is given by the diagonal matrix of eigenvalues:

$$Q(\mathbf{y}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$$

where $\mathbf{y}$ is the transformed vector in the basis of eigenvectors of A.