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Mathematics-Optional

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MECHANICS AND FLUID DYNAMICS 2013-2019

UPSC – MATHEMATICS optional – 2013 Questions

1. Prove that the necessary and sufficient condition that the vortex lines may be at right angles to the stream lines are

$u, v, w = \mu\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)$ where μ and φ are functions of x, y, z, t **[10M]**

2. Four solid spheres A, B, C and D each of mass m and radius a, are placed with their centres on the four corners of a square of side b. Calculate the moment of intertia of the system about a diagonal of the square. **[10M]**

3. If fluid fills the region of space on the positive side of the x-axis, which is a rigid boundary and if there be a source m at the point $(0, a)$ and an equal sink at $(0, b)$ and if the pressure on the negative side be the same as the pressure at infinity, show that the resultant pressure on the boundary is $\frac{\pi\rho m^2(a-b)^2}{\{2ab(a+b)\}}$ where ρ is the density of the fluid. **[15M]**

4. If n rectilinear vortices of the same strength K are symmetrically arranged as generators of a circular cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $\frac{8\pi^2a^3}{(n-1)K}$. Find the velocity at any point of the liquid. **[20M]**

UPSC – MATHEMATICS optional – 2014 Questions

1. Find the equation of motion of a compound pendulum using Hamilton's equations. **[10M]**

2. Given the velocity potential $\phi = \frac{1}{2}\log\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]$, determine the streamlines. **[20M]**

3. Find Navier-Stokes equation for a steady laminar flow of a viscous incompressible fluid between two infinite parallel plates. **[20M]**

UPSC – MATHEMATICS optional – 2015 Questions

1. Consider a uniform flow U_0 in the positive x-direction. A cylinder of radius a is located at the origin. Find the stream function and the velocity potential. Find also the stagnation points. **[10M]**

2. Calculate the moment of inertia of a solid uniform hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$ with mass m about the OZ-axis. **[10M]**

3. Solve the plane pendulum problem using the Hamiltonian approach and show that H is a constant of motion. **[15M]**

4. A Hamiltonian of a system with one degree of freedom has the form

$H = \frac{p^2}{2\alpha} - bqpe^{-\alpha t} + \frac{b\alpha}{2}q^2e^{-\alpha t}(\alpha + be^{-\alpha t}) + \frac{k}{2}q^2$ where α, b, k are constants, q is the generalized coordinate and p is the corresponding generalized momentum.

(i) Find a Lagrangian corresponding to this Hamiltonian. **[10M]**

(ii) Find an equivalent Lagrangian that is not explicitly dependent on time. **[10M]**

5. In an axisymmetric motion, show that stream function exists due to equation of continuity. Express the velocity components in terms of the stream function. Find the equation satisfied by the stream function if the flow is irrotational. **[20M]**

UPSC – MATHEMATICS optional – 2016 Questions

1. Does a fluid with velocity $\vec{q} = \left[z - \frac{2x}{r}, 2y - 3z - \frac{2y}{r}, x - 3y - \frac{2z}{r}\right]$ possess vorticity, where $\vec{q}(u, v, w)$ is the velocity in the Cartesian frame, $\vec{r} = (x, y, z)$ and $r^2 = x^2 + y^2 + z^2$? What is the circulation in the circle $x^2 + y^2 = 9, z = 0$? **[10M]**

2. Consider a single free particle of mass m, moving in space under no forces. If the particle starts from the origin at $t = 0$ and reaches the position (x, y, z) at time τ, find the Hamilton's characteristic function S as a function of x, y, z, τ. **[10M]**

3. A simple source of strength m is fixed at the origin O in a uniform stream of incompressible fluid moving with velocity U $\vec{\imath}$. Show that the velocity potential ϕ at any point P of the stream is $\frac{m}{r} - Ur\cos\theta$, where $OP = r \ and\ \theta$ is the angle which $\overrightarrow{OP}$ makes with the direction $\vec{\imath}$. Find the differential equation of the streamlines and show that they lie on the surfaces $Ur^2\sin^2\theta - 2m\cos\theta = constant$. **[15M]**

4. The space between two concentric spherical shells of radii a, $b(a < b)$ is filled with a liquid of density ρ. If the shells are set in motion, the inner one with velocity U in the x-direction and the outer one with velocity V in the y-direction, then show that the initial motion of the liquid is given by velocity potential $\phi = \frac{\left\{a^3U\left(1+\frac{1}{2}b^3r^{-3}\right)x - b^3V\left(1+\frac{1}{2}a^3r^{-3}\right)y\right\}}{(b^3-a^3)}$, where $r^2 = x^2 + y^2 + z^2$, the coordinates being rectangular. Evaluate the velocity at any point of the liquid. **[20M]**

5. A hoop with radius r is rolling, without slipping, down an inclined plane of length l and with angle of inclination ϕ. Assign appropriate generalized coordinates to the system. Determine the constraints, if any. Write down the Lagrangian equations for the system. Hence or otherwise determine the velocity of the hoop at the bottom of the inclined plane. **[15M]**

UPSC – MATHEMATICS optional – 2017 Questions

1. Show that the moment of inertia of an elliptic area of mass M and semi-axis a and b about a semi-diameter of length r is $\frac{1}{4}M\frac{a^2b^2}{r^2}$. Further, prove that the moment of inertia about a tangent is $\frac{5M}{4}p^2$, where p is the perpendicular distance from the centre of the ellipse to the tangent. **[10M]**

2. Two uniform rods AB, AC, each of mass m and length 2a, are smoothly hinged together at A and move on a horizontal plane. At time t, the mass centre of the rods is at the point (ξ, η) referred to fixed perpendicular axes Ox, Oy in the plane, and the rods make angles $\theta \pm \phi$ with Ox. Prove that the kinetic energy of the system is

$$m\left[\dot{\xi}^2 + \dot{\eta}^2 + \left(\frac{1}{3} + \sin^2\phi\right)a^2\,\dot{\theta}^2 + \left(\frac{1}{3} + \cos^2\phi\right)a^2\dot{\phi}^2\right].$$

Also derive Lagrange's equations of motion for the system if an external force with components $[\mathrm{x}, \mathrm{y}]$ along the axes acts at A. **[20M]**

3. A stream is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d. if V and v be the corresponding velocities of the stream and if the motion is assumed to be steady and diverging from the vertex of the cone, then prove that $\frac{v}{V} = \frac{D^2}{d^2}e^{(v^2-V^2)/2K}$ where K is the pressure divided by the density and is constant. **[15M]**

4. If the velocity of an incompressible fluid at the point (x, y, z) is given by $\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2-r^2}{r^5}\right)$, $r^2 = x^2 + y^2 + z^2$, then prove that the liquid motion is possible and that the velocity potential is $\frac{z}{r^3}$. Further, determine the streamlines. **[15M]**

UPSC – MATHEMATICS optional – 2018 Questions

1. For an incompressible fluid flow, two components of velocity (u, v, w) are given by $u = x^2 + 2y^2 + 3z^2$, $v = x^2y - y^2z + zx$. Determine the third component w so that they satisfy the equation of continuity. Also, find the z-component of acceleration. **[10M]**

2. Suppose the Lagrangian of a mechanical system is given by $L = \frac{1}{2}m(ax^2 + 2bxy + cy^2) - \frac{1}{2}k(ax^2 + 2bxy + cy^2)$, where $a, b, c, m(> 0), k(> 0)$ are constants and $b^2 \neq ac$. Write down the Lagrangian equations of motion and identify the system? **[20M]**

3. The Hamiltonian of a mechanical system is given by $H = p_1q_1 - aq_1^2 - bq_2^2 - p_2q_2$, where a, b are the constants. Solve the Hamiltonian equations and show that $\frac{p_2-bq_2}{q_1} = \text{constant}$. **[20M]**

4. For a two-dimensional potential flow, the velocity potential is given by $\phi = x^2y - xy^2 + \frac{1}{3}(x^3 - y^3)$. Determine the velocity components along the directions x and y. Also, determine the stream function ψ and check whether ϕ represents a possible case of flow or not. **[15M]**

UPSC – MATHEMATICS optional – 2019 Questions

1. A uniform rod OA, of length $2a$, free to turn about its end O, revolves with angular velocity ω about the vertical OZ through O, and is inclined at a constant angle α to OZ; find the value of α. **[10M]**

2. A circular cylinder of radius a and radius of gyration k rolls without slipping inside a fixed hollow cylinder of radius b. Show that the plane through axes moves in a circular pendulum of length $(b-a)\left(1+\frac{k^2}{a^2}\right)$ **[20M]**

3. Using Hamilton's equation, find the acceleration for a sphere rolling down a rough inclined plane, if x be the distance of the point of contact of the sphere from a fixed point on the plane. **[15M]**

4. A sphere of radius R, whose centre is at rest, vibrates radially in an infinite incompressible fluid of density ρ, which is at rest at infinity. If the pressure at infinity is Π, so that the pressure at the surface of the sphere at time t is $\Pi + \frac{1}{2}\rho\left\{\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right\}$. **[15M]**

5. Two sources, each of strength m, are placed at the points $(-a, 0), (a, 0)$ and a sink of strength $2m$ at origin. Show that the stream lines are the curves $(x^2+y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$, where λ is a variable parameter. Show also that the fluid speed at any point is $(2ma^2)/(r_1r_2r_3)$, where r_1, r_2 and r_3 are the distances of the points from the sources and the sink, respectively. **[20M]**