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## Online IAS Academy

# Mathematics-Optional

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COMPLEX ANALYSIS 2013-2019

# UPSC – MATHEMATICS optional – 2013 Questions

**1.** Prove that if $b\,e^{a+1} < 1$ where a and b are positive and real, then the function $z^n e^{-a} - be^z$ has n zeroes in the unit circle. **[10M]**

**2.** Using Cauchy's residue theorem, evaluate the integral

$I = \int_0^{\pi} sin^4\,\theta d\theta$ **[15M]**

# UPSC – MATHEMATICS optional – 2014 Questions

**1.** Prove that the function $f(z) = u + iv$, where

$f(z) = \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, z \neq 0; f(0) = 0$

Satisfies Cauchy-Riemann equations at the origin, but the derivative of $f$ at $z = 0$ does not exist. **[15M]**

**2.** Expand in Laurent series the function $f(z) = \frac{1}{z^2(z-1)}$ about $z = 0$ and $z = 1$. **[10M]**

**3.** Evaluate the integral $\int_0^{\pi} \frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2}$ using residues. **[20M]**

# UPSC – MATHEMATICS optional – 2015 Questions

**1.**Show that the function $v(x,y) = \ln(x^2 + y^2) + x + y$ is harmonic. Find its conjugate harmonic function $u(x,y)$. Also, find the corresponding analytic function $f(z) = u + iv$ in terms of z. **[10M]**

**2.**Find all possible Taylor's and Laurent's series expansions of the function $f(z) = \frac{2z-3}{z^2-3z+2}$ about the point $z = 0$. **[20M]**

**3.**State Cauchy's residue theorem. Using it, evaluate the integral $\int_C \frac{e^z+1}{z(z+1)(z-i)} dz; C: |z| = 2$

**[15M]**

# UPSC – MATHEMATICS optional – 2016 Questions

**1.** Is $v(x,y) = x^3 - 3xy^2 + 2y$ a harmonic function ? Prove your claim. If yes, find its conjugate harmonic function $u(x,y)$ and hence obtain the analytic function whose real and imaginary parts are $u$ *and* $v$ respectively. **[10M]**

**2.** Let $\gamma: [0,1] \to C$ *be the curve*

$$\gamma(t) = e^{2\pi it}, 0 \leq t \leq 1.$$

Find, giving justifications, the value of the contour integral $\int_\gamma \frac{dz}{4z^2-1}$ **[15M]**

**3.** Prove that every power series represents an analytic function inside its circle of convergence. **[20M]**

## UPSC – MATHEMATICS optional – 2017 Questions

**1.** Determine all entire functions $f(z)$ such that 0 is a removable singularity of $f\left(\frac{1}{z}\right)$. **[10M]**

**2.** Using contour integral method, prove that $\int_0^\infty \frac{x \sin mx}{a^2+x^2} dx = \frac{\pi}{2} e^{-ma}$. **[15M]**

## UPSC – MATHEMATICS optional – 2018 Questions

**1.** Prove that the function: $u(x,y) = (x-1)^3 - 3xy^2 + 3y^2$ is harmonic and find its harmonic conjugate and the corresponding analytic function $f(z)$ in terms of z. **[10M]**

**2.** Show by applying the residue theorem that $\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}, a > 0$. **[15M]**

**3.** Find the Laurent's series which represent the function $\frac{1}{(1+z^2)(z+2)}$ when

(i) $|z| < 1$

(ii) $1 < |z| < 2$

(iii) $|z| > 2$ **[15M]**

## UPSC – MATHEMATICS optional – 2019 Questions

**1.** Suppose $f(z)$ is analytic function on a domain $D$ in $\mathbb{C}$ and satisfies the equation $Im\, f(z) = (Re\, f(z))^2, Z \in D$. Show that $f(z)$ is constant in $D$. **[10M]**

**2.** Show that an isolated singular point $z_0$ of a function $f(z)$ is a pole of order $m$ if and only if $f(z)$ can be written in the form $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic and non zero at $z_0$. Moreover $\underset{z=z_0}{\text{Res}}\ f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if $m \geq 1$. **[15M]**

**3.** Evaluate the integral $\int_c Re(z^2)dz$ from $0$ to $2+4i$ along the curve $C$ where $C$ is a parabola $y = x^2$. **[10M]**

**4.** Obtain the first three terms of the Laurent series expansion of the function $f(z) = \frac{1}{(e^z-1)}$ about the point $z = 0$ valid in the region $0 < |z| < 2\pi$. **[10M]**