INDIANCIVILS.COM

Online IAS Academy

Mathematics-Optional

By Venkanna Sir and Satya Sir

Complex Analysis 2013 - 2021

UPSC – MATHEMATICS optional – 2013 Questions

1. Prove that if $b\,e^{a+1} < 1$ where a and b are positive and real, then the function $z^n e^{-a} - be^z$ has n zeroes in the unit circle. **[10M]**

2. Using Cauchy's residue theorem, evaluate the integral

$$I = \int_0^\pi sin^4\,\theta d\theta \qquad \text{[15M]}$$

UPSC – MATHEMATICS optional – 2014 Questions

1. Prove that the function $f(z) = u + iv$, where

$$f(z) = \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, z \neq 0; f(0) = 0$$

Satisfies Cauchy-Riemann equations at the origin, but the derivative of f at $z = 0$ does not

exist. **[15M]**

2. Expand in Laurent series the function $f(z) = \frac{1}{z^2(z-1)}$ about $z = 0$ and $z = 1$. **[10M]**

3. Evaluate the integral $\int_0^\pi \frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2}$ using residues. **[20M]**

UPSC – MATHEMATICS optional – 2015 Questions

1. Show that the function $v(x,y) = \ln(x^2 + y^2) + x + y$ is harmonic. Find its conjugate harmonic function $u(x,y)$. Also, find the corresponding analytic function $f(z) = u + iv$ in terms of z. **[10M]**

2. Find all possible Taylor's and Laurent's series expansions of the function $f(z) = \frac{2z-3}{z^2-3z+2}$ about the point $z = 0$. **[20M]**

3. State Cauchy's residue theorem. Using it, evaluate the integral $\int_c \frac{e^z+1}{z(z+1)(z-i)}dz; C: |z| = 2$

[15M]

UPSC – MATHEMATICS optional – 2016 Questions

1. Is $v(x,y) = x^3 - 3xy^2 + 2y$ a harmonic function? Prove your claim. If yes, find its conjugate harmonic function $u(x,y)$ and hence obtain the analytic function whose real and imaginary parts are u and v respectively. **[10M]**

2. Let $\gamma : [0,1] \to C$ be the curve

$$\gamma(t) = e^{2\pi it}, 0 \le t \le 1.$$

Find, giving justifications, the value of the contour integral $\int_\gamma \frac{dz}{4z^2-1}$ **[15M]**

3. Prove that every power series represents an analytic function inside its circle of convergence. **[20M]**

UPSC – MATHEMATICS optional – 2017 Questions

1. Determine all entire functions $f(z)$ such that 0 is a removable singularity of $f\left(\frac{1}{z}\right)$. **[10M]**

2. Using contour integral method, prove that $\int_0^\infty \frac{x \sin mx}{a^2+x^2} dx = \frac{\pi}{2} e^{-ma}$. **[15M]**

UPSC – MATHEMATICS optional – 2018 Questions

1. Prove that the function: $u(x,y) = (x-1)^3 - 3xy^2 + 3y^2$ is harmonic and find its harmonic conjugate and the corresponding analytic function $f(z)$ in terms of z. **[10M]**

2. Show by applying the residue theorem that $\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}, a > 0$. **[15M]**

3. Find the Laurent's series which represent the function $\frac{1}{(1+z^2)(z+2)}$ when

(i) $|z| < 1$

(ii) $1 < |z| < 2$

(iii) $|z| > 2$ **[15M]**

UPSC – MATHEMATICS optional – 2019 Questions

1. Suppose $f(z)$ is analytic function on a domain D in $\mathbb{C}$ and satisfies the equation $Im\, f(z) = (Re\, f(z))^2, Z \in D$. Show that $f(z)$ is constant in D. **[10M]**

2. Show that an isolated singular point z_0 of a function $f(z)$ is a pole of order m if and only if $f(z)$ can be written in the form $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic and non-zero at z_0. Moreover $\underset{z=z_0}{Res}\ f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if $m \geq 1$. **[15M]**

3. Evaluate the integral $\int_C Re(z^2)dz$ from 0 to $2 + 4i$ along the curve C where C is a parabola $y = x^2$. **[10M]**

4. Obtain the first three terms of the Laurent series expansion of the function $f(z) = \frac{1}{(e^z-1)}$ about the point $z = 0$ valid in the region $0 < |z| < 2\pi$. **[10M]**

UPSC – MATHEMATICS optional – 2020 Questions

1. Evaluate the integral $\int_C (z^2 + 3z)dz$ counterclockwise from $(2,0)$ to $(0,2)$ along the curve C, where C is the circle $|z| = 2$. **[10M]**

2. Using Contour integration, evaluate the integral $\int_0^{2\pi} \frac{1}{3+2\sin\theta}d\theta$. **[20M]**

3. If $v(r,\theta) = \left(r - \frac{1}{r}\right)\sin\theta, \ r \neq 0$, then find an analytic function $f(z) = u(r,\theta) + iv(r,\theta)$ **[15M]**

UPSC – MATHEMATICS optional – 2021 Questions

1. Let $c: [0,1] \to \mathbb{C}, c(t) = e^{4\pi it}, 0 \leq t \leq 1$. Evaluate the contour integral $\int_c \frac{dz}{2z^2-5z+2}$ **[10M]**

2. Find the Laurent series expansion of $f(z) = \frac{z^2-z+1}{z(z^2-3z+2)}$ in the powers of $(z+1)$ in the region $|z| > 3$. **[20M]**

3. Let f be an entire function whose Taylor series expansion with centre $z = 0$ has infinitely many terms. Show that $z = 0$ is an essential singularity of $f\left(\frac{1}{z}\right)$. **[15M]**

4. Using contour integration, evaluate the integral $\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x(x^2+a^2)}, a > 0$. **[20M]**