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# **Mathematics-Optional**

**By Venkanna Sir and Satya Sir**

**Abstract Algebra 2013 - 2021**

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## UPSC – MATHEMATICS optional – 2013 Questions

1. Show that the set of matrices  $S = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in R \right\}$  is a field under the usual binary operations of matrix addition and matrix multiplication. What are the additive and multiplicative identities and what is the inverse of  $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ ? Consider the map  $f: C \rightarrow S$  defined by  $f(a + ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ . Show that  $f$  is an isomorphism. (Here  $R$  is the set of real numbers and  $C$  is the set of complex numbers. [10M]

2. Give an example of an infinite group in which every element has finite order. [10M]

3. What are the orders of the following permutations in  $S_{10}$ ?

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 8 & 7 & 3 & 10 & 5 & 4 & 2 & 6 & 9 \end{pmatrix} \text{ and } (1\ 2\ 3\ 4\ 5)(6\ 7). \quad [10M]$$

4. What is the maximal possible order of an element in  $S_{10}$ ? Why? Give an example of such an element. How many elements will there be in  $S_{10}$  of that order? [13M]

5. Let  $J = \{a + bi \mid a, b \in \mathbb{Z}\}$  be the ring of Gaussian integers (subring of  $\mathbb{C}$ ). Which of the following is  $J$ : Euclidean domain, principal ideal domain, unique factorization domain? Justify your answer. [15M]

6. Let  $R^C$  = ring of all real valued continuous functions on  $[0, 1]$ , under the operations.

$$(f + g)x = f(x) + g(x)$$

$$(fg)x = f(x)g(x).$$

$$\text{Let } M = \left\{ f \in R^C \mid f\left(\frac{1}{2}\right) = 0 \right\}.$$

Is  $M$  a maximal ideal of  $R$ ? Justify your answer. [15M]

## UPSC – MATHEMATICS optional – 2014 Questions

1. Let  $G$  be the set of all real  $2 \times 2$  matrices  $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ , where  $xz \neq 0$ . Show that  $G$  is a group under matrix multiplication. Let  $N$  denote the subset  $\left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : a \in R \right\}$ . Is  $N$  a normal subgroup of  $G$ ? Justify your answer. [10M]

2. Show that  $\mathbb{Z}_7$  is a field. Then find  $(|5| + |6|)^{-1}$  and  $(-|4|)^{-1}$  in  $\mathbb{Z}_7$ . [15M]

3. Show that the set  $\{a + b\omega : \omega^3 = 1\}$ , where  $a$  and  $b$  are real numbers, is a field with respect to usual addition and multiplication. [15M]

4. Prove that the set  $Q(\sqrt{5}) = \{a + b\sqrt{5} : a, b \in Q\}$  is a commutative ring with identity.

[15M]

### UPSC – MATHEMATICS optional – 2015 Questions

1. (i) How many generators are there of the cyclic group  $G$  of order 8? Explain.  
(ii) Taking a group  $\{e, a, b, c\}$  of order 4, where  $e$  is the identity, construct composition tables showing that one is cyclic while the other is not. [10M]
2. Give an example of a ring having identity but a subring of this having a different identity. [10M]
3. If  $R$  is a ring with unit element 1 and  $\phi$  is homomorphism of  $R$  onto  $R'$ , prove that  $\phi(1)$  is the unit element of  $R'$ . [15M]
4. Do the following sets form integral domains with respect to ordinary addition and multiplication? If so, state if they are fields :
- (i) The set of numbers of the form  $b\sqrt{2}$  with  $b$  rational  
(ii) The set of even integers  
(iii) The set of positive integers [15M]

### UPSC – MATHEMATICS optional – 2016 Questions

1. Let  $K$  be a field and  $K[X]$  be the ring of polynomials over  $K$  in a single variable  $X$ . For a polynomial  $f \in K[X]$ , let  $(f)$  denote the ideal in  $K[X]$  generated by  $f$ . Show that  $(f)$  is a maximal ideal in  $K[X]$  if and only if  $f$  is an irreducible polynomial over  $K$ . [10M]
2. Let  $p$  be a prime number and  $Z_p$  denote the additive group of integers modulo  $p$ . Show that every non-zero element of  $Z_p$  generates  $Z_p$ . [15M]
3. Let  $K$  be an extension of a field  $F$ . Prove that the elements of  $K$ , which are algebraic over  $F$ , form a subfield of  $K$ . Further, if  $F \subset K \subset L$  are fields,  $L$  is algebraic over  $K$  and  $K$  is algebraic over  $F$ , then prove that  $L$  is algebraic over  $F$ . [20M]
4. Show that every algebraically closed field is infinite. [15M]

## UPSC – MATHEMATICS optional – 2017 Questions

1. Let  $F$  be a field and  $F[X]$  denote the ring of polynomials over  $F$  in a single variable  $X$ . For  $f(X), g(X) \in F[X]$  with  $g(X) \neq 0$ , show that there exist  $q(X), r(X) \in F[X]$  such that  $\deg(r(X)) < \deg(g(X))$  and  $f(X) = q(X) \cdot g(X) + r(X)$ . [20M]
2. Let  $G$  be a group of order  $n$ . Show that  $G$  is isomorphic to a subgroup of the permutation group  $S_n$ . [10M]
3. Show that the groups  $Z_5 \times Z_7$  and  $Z_{35}$  are isomorphic. [15M]
4. Let  $R$  be an Integral domain with unit element. Show that any unit in  $R[x]$  is a unit in  $R$ . [10M]

## UPSC – MATHEMATICS optional – 2018 Questions

1. Show that the quotient group of  $(\mathbb{R}, +)$  modulo  $\mathbb{Z}$  is isomorphic to the multiplicative group of complex numbers on the unit circle in the complex plane. Here  $\mathbb{R}$  is the set of real numbers and  $\mathbb{Z}$  is the set of integers. [15M]
2. Find all the proper subgroups of the multiplicative group of the field  $(Z_{13}, +_{13}, \times_{13})$ , where  $+_{13}$  and  $\times_{13}$  represent addition modulo 13 and multiplication modulo 13 respectively. [15M]

## UPSC – MATHEMATICS optional – 2019 Questions

1. Let  $G$  be a finite group,  $H$  and  $K$  subgroups of  $G$  such that  $K \subset H$ . Show that  $(G:K) = (G:H)(H:K)$ . [10M]
2. If  $G$  and  $H$  are finite groups whose orders are relatively prime, then prove that there is only one homomorphism from  $G$  to  $H$ , the trivial one. [10M]
3. Write down all quotient groups of the group  $Z_{12}$ . [10M]
4. Let  $a$  be an irreducible element of the Euclidean ring  $R$ , then prove that  $R/(a)$  is a field. [10M]

## UPSC – MATHEMATICS optional – 2020 Questions

1. Let  $S_3$  and  $Z_3$  be permutation group on 3 symbols and group of residue classes module 3 respectively. Show that there is no homomorphism of  $S_3$  in  $Z_3$  except the trivial homomorphism. [10M]
2. Let  $R$  be a principal ideal domain. Show that every ideal of a quotient ring of  $R$  is principal ideal and  $R/P$  is a principal ideal domain for a prime ideal  $P$  and  $R$ . [10M]

3. Let  $G$  be a finite cyclic group of order  $n$ . Then prove that  $G$  has  $\phi(n)$  generators (where  $\phi$  is Euler's  $\phi$ -function). [15M]
4. Let  $R$  be a finite field of characteristic  $p(> 0)$ . Show that the mapping  $f: R \rightarrow R$  defined by  $f(a) = a^p, \forall a \in R$  is an isomorphism. [15M]

### UPSC – MATHEMATICS optional – 2021 Questions

1. Let  $m_1, m_2, \dots, m_k$  be positive integers and  $d > 0$  the greatest common divisor of  $m_1, m_2, \dots, m_k$ . Show that there exist integers  $x_1, x_2, \dots, x_k$  such that  $d = x_1 m_1 + x_2 m_2 + \dots + x_k m_k$  [10M]
2. Let  $F$  be a field and  $f(x) \in F[x]$  a polynomial of degree  $> 0$  over  $F$ . Show that there is a field  $F'$  and an imbedding  $q: F \rightarrow F'$  show that the polynomial  $f^q \in F'[x]$  has a root in  $F'$ , where  $f^q$  is obtained by replacing each coefficient  $a$  of  $f$  by  $q(a)$ . [15M]
3. Show that there are infinitely many subgroups of the additive group  $Q$  of rational numbers. [15M]