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Mathematics-Optional

By Venkanna Sir and Satya Sir

Calculus 2013 - 2021

UPSC – MATHEMATICS optional – 2013 Questions

1. Evaluate $\int_0^1 \left(2x \sin\frac{1}{x} - \cos\frac{1}{x}\right) dx$. **[10M]**

2. Using Lagrange's multiplier method, find the shortest distance between the line $y = 10 - 2x$ and ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$. **[20M]**

3. Compute $f_{xy}(0,0)$ *and* $f_{yx}(0,0)$ for the function

$$f(x,y) = \begin{cases} \frac{xy^3}{x+y^2}, (x,y) \neq (0,0) \\ 0 \ \ , (x,y) = (0,0). \end{cases}$$

Also, discuss the continuity of f_{xy} *and* f_{yx} at $(0,0)$. **[15M]**

4. Evaluate $\iint_D xy\, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$. **[15M]**

UPSC – MATHEMATICS optional – 2014 Questions

1. Prove that between two real roots of $e^x \cos x + 1 = 0$, a real root of $e^x \sin x + 1 = 0$ lies. **[10M]**

3. Evaluate $\int_0^1 \frac{\log_e(1+x)}{1+x^2}\, dx$ **[10M]**

4. By using the transformation $x + y = u,\ y = uv$, evaluate the integral $\iint\{xy(1-x-y)\}^{1/2} dxdy$ taken over the area enclosed by the straight lines $x = 0, y = 0$ *and* $x + y = 1$. **[15M]**

5. Find the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a. **[15M]**

6. Find the maximum or minimum values of $x^2 + y^2 + z^2$ subject to the conditions $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$. Interpret the result geometrically. **[20M]**

UPSC – MATHEMATICS optional – 2015 Questions

1. Evaluate the following limit: $\lim_{x \to a} \left[2 - \frac{x}{a}\right]^{\tan\left(\frac{\pi x}{2a}\right)}$ **[10M]**

2. Evaluate the following integral: $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x} + \sqrt[3]{\cos x}} dx$. **[10M]**

3. A conical tent is of given capacity. For the least amount of Canvas required, for it, find the ratio of its height to the radius of its base. **[13M]**

4. Evaluate the integral $\iint_R (x-y)^2 cos^2(x+y)dxdy$ where R is the rhombus with successive vertices as $(\pi,0)(2\pi,\pi)(\pi,2\pi)(0,\pi)$. **[12M]**

5. Evaluate $\iint_R \sqrt{|y-x^2|}dxdy$ where $R = \{-1,1\,;0,2]$. **[13M]**

6. For the function $f(x,y) = \begin{cases} \frac{x^2-x\sqrt{y}}{x^2+y}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Examine the continuity and differentiability. **[12M]**

UPSC – MATHEMATICS optional – 2016 Questions

1. Evaluate $I = \int_0^1 \sqrt[3]{x \log\left(\frac{1}{x}\right)}\, dx$ **[10M]**

2. Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the conditions $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$ and $x+y-z+0$. **[20M]**

3. Let $f(x,y) = \begin{cases} \frac{2x^4y-5x^2y^2+y^5}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Find a $\delta > 0$ such that $|f(x,y) - f(0,0)| < .01$, whenever $\sqrt{x^2+y^2} < \delta$. **[15M]**

4. Find the surface area of the plane $x+2y+2z = 12$ cut off by $x=0, y=0$ and $x^2+y^2 = 16$. **[15M]**

5. Evaluate $\iint_R f(x,y)dx\,dy$ over the rectangle $R = [0,1\,;0,1]$ where

$f(x,y) = \begin{cases} x+y, & \text{if } x^2 < y < 2x^2 \\ 0, & \text{elsewhere} \end{cases}$ **[15M]**

UPSC – MATHEMATICS optional – 2017 Questions

1. Integrate the function $f(x,y) = xy(x^2+y^2)$ over the domain $R: \{-3 \leq x^2 - y^2 \leq 3, 1 \leq xy \leq 4\}$ **[10M]**

2. Find the volume of the solid above the *xy*-palane and directly below the portion of the elliptic paraboloid $x^2 + \frac{y^2}{4} = z$ which is cut off by the palne $z = 9$. **[15M]**

3. If $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Calculate $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ at $(0,0)$. **[15M]**

4. Examine if the improper integral $\int_0^3 \frac{2xdx}{(1-x^2)^{2/3}}$ exists. **[10M]**

5. Prove that $\frac{\pi}{3} \leq \iint_D \frac{dxdy}{\sqrt{x^2+(y-2)^2}} \leq \pi$ where D is the unit disc. **[10M]**

UPSC – MATHEMATICS optional – 2018 Questions

1. Determine if $\lim_{z\to 1}(1-z)\tan\frac{\pi z}{2}$ exists or not. If the limit exists, then find its value. **[10M]**

2. Find the limit **[10M]**

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{r=0}^{n-1}\sqrt{n^2-r^2}.$$

3. Find the shortest distance from the point $(1,0)$ to the parabola $y^2 = 4x$. **[13M]**

4. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ revolves about the x-axis. Find the volume of the solid of revolution. **[13M]**

5. Find the shortest distance between the lines

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$

and the z-axis. **[12M]**

6. Let $f(x,y) = xy^2$, if $y > 0$

$= -xy^2$, If $y \leq 0$

Determine which of $\frac{\partial f}{\partial x}(0,1)$ and $\frac{\partial f}{\partial y}(0,1)$ exists and which does not exist. **[12M]**

7. Find the maximum and the minimum values of $x^4 - 5x^2 + 4$ on the interval $[2,3]$. **[13M]**

8. Evaluate the integral $\int_0^a \int_{x/a}^x \frac{x\,dydx}{x^2+y^2}$. **[12M]**

UPSC – MATHEMATICS optional – 2019 Questions

1. Let $f: \left[0,\frac{\pi}{2}\right] \to R$ be a continuous function such that $f(x) = \frac{\cos^2 x}{4x^2-\pi^2}$, $0 \leq x < \frac{\pi}{2}$

Find the value of $f\left(\frac{\pi}{2}\right)$ **[10M]**

2. Let $f: D(\subseteq R^2) \to R$ be a function and $(a,b) \in D$. If $f(x,y)$ is continuous at (a,b), then show that the functions $f(x,b)$ and $f(a,y)$ are continuous at $x = a$ and at $y = b$ respectively. **[10M]**

3. Is $f(x) = |\cos x| + |\sin x|$ differentiable at $x = \frac{\pi}{2}$? If yes, then find its derivative at $x = \frac{\pi}{2}$. If no, then give a proof of it. **[10M]**

4. Find the maximum and the minimum value of the function $f(x) = 2x^3 - 9x^2 + 12x + 6$ on the interval $[2, 3]$. **[15M]**

5. If $u = \sin^{-1}\sqrt{\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}$ then show that $\sin^2 u$ is a homogeneous function of x and y of degree $-\frac{1}{6}$. Hence show that $x^2\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial^2}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right)$ **[12M]**

6. Using the Jabobian method, show that if $f'(x) = \frac{1}{1+x^2}$ and $f(0) = 0$, then

$f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$ **[08M]**

UPSC – MATHEMATICS optional – 2020 Questions

1. Evaluate $\sum_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x}$. **[10M]**

2. Find all the asymptotes of the curve $(2x + 3)y = (x - 1)^2$. **[10M]**

3. Evaluate $\int_0^1 tan^{-1}\left[1 - \frac{1}{x}\right]dx$. **[15M]**

4. Consider the function $f(x) = \int_0^x (t^2 - 5t + 4)(t^2 - 5t + 6)dt$.

(i) Find the critical points of the function $f(x)$.

(ii) Find the points at which local minimum occurs.

(iii) Find the points at which local maximum occurs.

(iv) Find the number of zeros of the function $f(x)$ in $[0, 5]$. **[20M]**

5. Find an extreme value of the function $u = x^2 + y^2 + z^2$, subject to the condition $2x + 3y + 5z = 30$, by using Lagrange's method of undetermined multiplier. **[20M]**

UPSC – MATHEMATICS optional – 2021 Questions

1. Given: $\Delta(x) = \begin{vmatrix} f(x+\alpha) & f(x+2\alpha) & f(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}$ **[10M]**

Where f is a real valued differentiable function and α is a constant. Find $\lim_{x\to 0}\frac{\Delta(x)}{x}$.

2. Show that between any two roots of $e^x \cos x = 1$, there exists at least one root of $e^x \sin x - 1 = 0$. **[10M]**

3. Given that $f(x, y) = |x^2 - y^2|$. Find $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$. Hence show that $f_{xy}(0, 0) = f_{yx}(0, 0)$. **[15M]**

4. If $u = x^2 + y^2,\ v = x^2 - y^2$, where $x = r\cos\theta$, $y = r\sin\theta$, then find $\frac{\partial(u,v)}{\partial(r,\theta)}$. **[07M]**

5. If $\int_0^x f(t)dt = x + \int_x^1 t\, f(t)dt$, then find the value of $f(1)$. **[05M]**

6. Express $\int_a^b (x-a)^m (b-x)^n dx$ in terms of Beta function. **[08M]**

7. Show that the entire area of the Astroid : $x^{2/3} + y^{2/3} = a^{2/3}$ is $\frac{3}{8}\pi a^2$. **[15M]**