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Mathematics-Optional

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Linear Alegebra 2013 - 2021

UPSC – MATHEMATICS optional – 2013 Questions

1. Find the inverse of the matrix:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 7 \\ 3 & 2 & -1 \end{bmatrix}$$

by using elementary row operations. Hence solve the system of linear equations

$$\begin{aligned} x + 3y + z &= 10 \\ 2x - y + 7z &= 21 \\ 3x + 2y - z &= 4 \end{aligned}$$

[10M]

2. Let A be a square matrix and A^* be its adjoint, show that the eigenvalues of matrices AA^* and A^*A are real. Further show that trace $(AA^*) = trace\ (A^*A)$. **[10M]**

3. Let P_n denote the vector space of all real polynomials of degree atmost n and $T: P_2 \to P_3$ be a linear transformation given by $T(p(x)) = \int_o^x p(t)dt, \quad p(x) \in P_2$.
Find the matrix of T with respect to the bases $\{1, x, x^2\}$ and $\{1, x, 1 + x^2, 1 + x^3\}$ of P_2 and P_3 respectively. Also, find the null space of T. **[10M]**

4. Let V be an n-dimensional vector space and $T: V \to V$ be an invertible linear operator. If $\beta = \{X_1, X_2, \dots, X_n\}$ is a basis of V, show that $\beta' = \{TX_1, TX_2, \dots, TX_n\}$ is also a basis of V. **[08M]**

5. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$ where $\omega (\neq 1)$ is a cube root of unity. If $\lambda_1, \lambda_2, \lambda_3$ denote the eigenvalues of A^2, show that $|\lambda_1| + |\lambda_2| + |\lambda_3| \leq 9$. **[08M]**

6. Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 8 & 12 \\ 3 & 5 & 8 & 12 & 17 \\ 5 & 8 & 12 & 17 & 23 \\ 8 & 12 & 17 & 23 & 30 \end{bmatrix}$ **[08M]**

7. Let A be a Hermetian matrix having all distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. If $X_1, X_2, \dots, X_n$ are corresponding eigenvectors then show that the $n \times n$ matrix C whose k^{th} column consists of the vector X_k is non-singular. **[08M]**

8. Show that the vectors $X_1 = (1, 1 + i), X_2 = (i, -i, 1 - i)$ and $X_3 = (0,\ 1 - 2i, 2 - i)$ in C^3 are linearly independent over the field of real numbers but are linearly dependent over the field of complex numbers. **[08M]**

UPSC – MATHEMATICS optional – 2014 Questions

1. Find one vector in R^3 which generates the intersection of V and W, where V is the xy plane and W is the space generated by the vectors $(1, 2, 3)$ *and* $(1, -1, 1)$. **[10M]**

2. Using elementary row or column operations, find the rank of the matrix **[10M]**

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

3. Let V and W be the following subspaces of R^4:

$V = \{(a, b, c, d): b - 2c + d = 0\}$ and $W = \{(a, b, c, d): a = d, b = 2c\}$

Find a basis and the dimension of (i) V, (ii) W, (iii) $V \cap W$. **[15M]**

4. Investigate the values of λ and μ so that the equations $x + y + z = 6$, $x + 2y + 3z = 10, x + 2y + \lambda z = \mu$ have (1) no solution, (2) a unique solution, (3) an infinite number of solutions. **[10M]**

5. Verify Cayley – Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and hence find its inverse. Also, find the matrix represented by $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10\,I$. **[10M]**

6. Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. Find the eigen values of A and the corresponding eigen vectors. **[08M]**

7. Prove that the eigen values of a unitary matrix have absolute value 1. **[07M]**

UPSC – MATHEMATICS optional – 2015 Questions

1. The vectors $V_1 = (1, 1, 2, 4), V_2 = (2, -1, -5, 2), V_3 = (1, -1, -4, 0)$ and $V_6 = (2, 1, 1, 6)$ are linearly independent. Is it true? Justify your answer. **[10M]**

2. Reduce the following matrix to row echelon form and hence find its rank:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$$

[10M]

3. If matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then find A^{30}. **[12M]**

4. Find the eigen values and eigen vectors of the matrix:

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$ **[12M]**

5. Let $\mathrm{V} = \mathrm{R}^3$ and $\mathrm{T} \in \mathrm{A(V)}$, for all $a_1 \in \mathrm{A(V)}$, be defined by **[12M]**

$\mathrm{T}(a_1, a_2, a_3) = (2a_1 + 5a_2 + a_3, -3a_1 + a_2 - a_3, -a_1 + 2a_2 + 3a_3)$ What is the matrix T relative to the basis $\mathrm{V}_1 = (1, 0, 1)$ $\mathrm{V}_2 = (-1, 2, 1)$ $\mathrm{V}_3 = (3, -1, 1)$?

6. Find the dimension of the subspace of R^4, spanned by the set $\{(1, 0, 0), (0, 1, 0, 0), (1, 2, 0, 1), \ (0, 0, 0, 1)\}$ Hence find its basis. **[12M]**

UPSC – MATHEMATICS optional – 2016 Questions

1. Using elementary row operations, find the inverse of $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ **[06M]**

2. If $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$, then find $A^{14} + 3A - 2I$. **[04M]**

3. (i) Using elementary row operations, find the condition that the linear equations

$$\begin{aligned} x - 2y + z &= a \\ 2x + 7y - 3z &= b \\ 3x + 5y - 2z &= c \end{aligned}$$ have a solution. **[07M]**

(ii) If $W_1 = \{(x, y, z) | x + y - z = 0\}$

$W_2 = \{(x, y, z) | 3x + y - 2z = 0\}$

$W_3 = \{(x, y, z) | x - 7y + 3z = 0\}$ then find $\dim(\mathrm{W}_1 \cap \mathrm{W}_2 \cap \mathrm{W}_3)$ and $\dim(\mathrm{W}_1 + \mathrm{W}_2.)$ **[03M]**

4. (i) If $M_2(R)$ is space of real matrices of order 2×2 and $P_2(x)$ is the space of real polynomials of degree at most 2, then find the matrix representation of $T: M_2(R) \to P_2(x)$, such that $T\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = a + c + (a - d)x + (b + c)x^2$, with respect to the standard bases of $M_2(R)\ and\ P_2(x)$. Further find the null space of T. **[10M]**

(ii) If $T: P_2(x) \to P_3(x)$ is such that $T(f(x) = f(x) + 5\int_0^x f(t)dt$, then choosing $\{1, 1 + x, 1 - x^2\}$ and $\{1, x, x^2, x^3\}$ as bases of $P_2(x)$ and $P_3(x)$ respectively, find the matrix of T. **[06M]**

5. If $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then find the eigenvalues and eigenvectors of A. **[08M]**

6. Prove that eigenvalues of a Hermitian matrix are all real. **[08M]**

7. If $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$ is the matrix representation of a linear transformation $T: P_2(x) \to P_2(x)$ with respect to the bases $\{1-x, x(1-x), x(1+x)\}$ and $\{1, 1+x, 1+x^2\}$, then find T. **[18M]**

UPSC – MATHEMATICS optional – 2017 Questions

1. Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$. Find a non-singular matrix P such that $P^{-1}AP$ is a diagonal matrix.

[10M]

2. Show that similar matrices have the same characteristic polynomial. **[10M]**

3. Suppose U and W are distinct four dimensional subspaces of a vector space V, where $\dim V = 6$. Find the possible dimensions of subspace $U \cap W$. **[10M]**

4. Consider the matrix mapping $A: R^4 \to R^3$, where $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix}$. Find a basis and dimension of the image of A and those of the kernel A. **[15M]**

5. Prove that distinct non-zero eigenvectors of a matrix are linearly independent.

[10M]

6. Consider the following system of equations in x, y, z: **[15M]**

$$x + 2y + 2z = 1$$
$$x + ay + 3z = 3$$
$$x + 11y + az = b.$$

(i) For which values of a does the system have a unique solution?

(ii) For which pair of values (a, b) does the system have more than one solution?

UPSC – MATHEMATICS optional – 2018 Questions

1. Let A be a 3×2 matrix and B a 2×3 matrix. Show that $C = A \cdot B$ is a singular matrix. **[10M]**

2. Express basis vectors $e_1 = (1,0)$ *and* $e_2 = (0,1)$ as linear combinations of $\alpha_1 = (2,-1)$ and $\alpha_2 = (1,3)$. **[10M]**

3. Show that if A and B are similar $n \times n$ matrices, then they have same eigenvalues. **[12M]**

4. For the system of linear equations **[13M]**

$$x + 3y - 2z = -1$$

$$5y + 3z = -8$$

$$x - 2y - 5z = 7$$

Determine which of the following statements are true and which are false:

(i) The system has no solution.

(ii) The system has a unique solution.

(iii) The system has infinitely many solutions.

UPSC – MATHEMATICS optional – 2019 Questions

1. Let $T: R^2 \longrightarrow R^2$ be a linear map such that $T(2,1) = (5,7)$ and $T(1,2) = (3,3)$. If A is the matrix corresponding to T with respect to the standard bases e_1, e_2, then find Rank (A). **[10M]**

2. If $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$ then show that $AB = 6I_3$. Use this result to solve the following system of equations:

$$\begin{aligned} 2x + y + z &= 5 \\ x - y &= 0 \\ 2x + y - z &= 1 \end{aligned}$$

[10M]

3. Let A and B be two orthogonal matrices of same order and $\det A + \det B = 0$. Show that $A + B$ is a singular matrix. **[15M]**

4. Let $A = \begin{bmatrix} 5 & 7 & 2 & 1 \\ 1 & 1 & -8 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & -3 & 1 \end{bmatrix}$

(i) Find the rank of matrix A.

(ii) Find the dimension of the subspace

$$V = \left\{ (x_1, x_2, x_3, x_4) \in R^4 \middle| A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right\}$$ **[15+5=20M]**

5. State the Cayley-Hamilton theorem. Use this theorem to find A^{100}, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ **[15M]**

UPSC – MATHEMATICS optional – 2020 Questions

1. Consider the set V of all $n \times n$ real magic squares. Show that V is a vector space over R. give examples of two distinct 2×2 magic squares. **[10M]**

2. Let $M_2(R)$ be the vector space of all 2×2 real matrices. Let $B = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}$ Suppose $T: M_2(R) \to M_2(R)$ is a linear transformation defined by $T(A) = BA$. Find the rank and nullity of T. Find a matrix A which maps to the null matrix. **[10M]**

3. Define an $n \times n$ matrix as $A = I - 2u \cdot u^T$, where u is a unit column vector. **[20M]**

(i) Examine if A is symmetric.

(ii) Examine if A is orthogonal.

(iii) Show that trace $(A) = n - 2$.

(iv) Find $A_{3\times3}$, when $u = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$

4. Let F be a subfield of complex numbers and T a function from $F^3 \to F^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2 + 3x_3, \; -3x_1 + x_2 - x_3)$. What are the conditions on a, b, c such that (a, b, c) be in the null space of T? Find the nullity of T. **[15M]**

5. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$ **[15M]**

(i) Find AB

(ii) Find $\det(A)$ and $\det(B)$.

(iii) Solve the following system of linear equations:

$x + 2z = 3,\ 2x - y + 3z = 3,\ 4x + y + 8z = 14$

UPSC – MATHEMATICS optional – 2021 Questions

1. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then show that $A^2 = A^{-1}$ (without finding A^{-1}) **[10M]**

2. Find the matrix associated with the linear operator on $V_3(R)$ defined by $T(a, b, c) = (a + b, a - b, 2c)$ with respect to the ordered basis. $B = \{(0, 1, 1),\ (1, 0, 1), (1, 1, 0)\}$ **[10M]**

3. Show that $S = \{(x, 2y, 3x) : x, y \text{ are real numbers}\}$ is a subspace of $R^3(R)$. Find two bases of S. Also find the dimension of S. **[15M]**

4. Prove that the eigen vectors, corresponding to two distinct eigen values of a real symmetric matrix, are orthogonal. **[08M]**

5. For two square matrices A and B of order 2, show that trace $(AB) =$ trace (BA). Hence show that $AB - BA \neq I_2$ is an identity matrix of order 2. **[07M]**

6. Reduce the following matrix to a row-reduced echelon form and hence also, find its rank:

$A = \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{bmatrix}$ **[10M]**

7. Find the eigen values and the corresponding eigen vectors of the matrix $A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, over the complex-number field. **[10M]**